ECE351 HW#5: Solutions

2.44. If $y(t) = x(t) * h(t)$ is the output of an LTI system with input $x(t)$ and impulse response $h(t)$, then show that

$$\frac{d}{dt}y(t) = x(t) * \left(\frac{d}{dt}h(t)\right)$$

and

$$\frac{d}{dt}y(t) = \left(\frac{d}{dt}x(t)\right) * h(t)$$

$$\frac{d}{dt}y(t) = \frac{d}{dt}x(t) * h(t)$$
$$= \frac{d}{dt}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Assuming that the functions are sufficiently smooth,

the derivative can be pulled throught the integral

$$\frac{d}{dt}y(t) = \int_{-\infty}^{\infty} x(\tau)\frac{d}{dt}h(t-\tau)d\tau$$

Since $x(\tau)$ is independent of t

$$\frac{d}{dt}y(t) = x(t) * \left(\frac{d}{dt}h(t)\right)$$

The convolution integral can also be written as

$$\frac{d}{dt}y(t) = \frac{d}{dt}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$
$$= \int_{-\infty}^{\infty} h(\tau)\frac{d}{dt}x(t-\tau)d\tau$$
$$= \left(\frac{d}{dt}x(t)\right) * h(t)$$

2.50. Evaluate the step response for the LTI systems represented by the following impulse responses:
(a) $h[n] = (-1/2)^n u[n]$

Solution:

When the input is $u[n]$,

the output $s[n] = \sum_{k=-\infty}^{\infty} (-1/2)^k u[k]u[n-k]$.

for $n < 0$

$$s[n] = 0$$

for $n \geq 0$

$$s[n] = \sum_{k=0}^{n} (-\frac{1}{2})^k$$

$$s[n] = \frac{1}{3}\left(2 + \left(-\frac{1}{2}\right)^n\right)$$

$$s[n] = \begin{cases} \frac{1}{3}\left(2 + (-\frac{1}{2})^n\right) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

(b) $h[n] = \delta[n] - \delta[n-2]$

Solution:

When the input is $u[n]$,

the output $s[n] = \sum_{k=-\infty}^{\infty}(\delta[k]-\delta[k-2])u[n-k]$.

for $n < 0$, $s[n] = 0$;

for $n = 0$ or 1, $s[n] = 1$;

for $n > 1$, $s[n] = 0$.

(e) $h(t) = e^{-|t|}$

for $t < 0$

$$s(t) = \int_{-\infty}^{t} e^{\tau}\, d\tau = e^{t}$$

for $t \geq 0$

$$s(t) = \int_{-\infty}^{0} e^{\tau}\, d\tau + \int_{0}^{t} e^{-\tau}\, d\tau = 2 - e^{-t}$$

$$s(t) = \begin{cases} e^{t} & t < 0 \\ 2 - e^{-t} & t \geq 0 \end{cases}$$

(g) $h(t) = (1/4)(u(t) - u(t-4))$

for $t < 0$

$$s(t) = 0$$

for $t < 4$

$$s(t) = \frac{1}{4}\int_{0}^{t} d\tau = \frac{1}{4}t$$

for $t \geq 4$

$$s(t) = \frac{1}{4}\int_{0}^{4} d\tau = 1$$

$$s(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{4}t & 0 \leq t < 4 \\ 1 & t \geq 4 \end{cases}$$

2.57. Determine the output of the systems described by the following differential equations with input andinitial conditions as specified:

(a) $\frac{d}{dt}y(t) + 10y(t) = 2x(t), \quad y(0^-) = 1, x(t) = u(t)$

$$\begin{aligned}
t \geq 0 & \quad \text{natural: characteristic equation} \\
r + 10 &= 0 \\
r &= -10 \\
y^{(n)}(t) &= ce^{-10t} \\
& \quad \text{particular} \\
y^{(p)}(t) &= ku(t) = \frac{1}{5}u(t) \\
y(t) &= \frac{1}{5} + ce^{-10t} \\
y(0^-) = 1 &= \frac{1}{5} + c \\
c &= \frac{4}{5} \\
y(t) &= \frac{1}{5}\left[1 + 4e^{-10t}\right] u(t)
\end{aligned}$$

2.59. Determine the output of the systems described by the following difference equations with input and initial conditions as specified:

(a) $y[n] - \frac{1}{2}y[n-1] = 2x[n], \quad y[-1] = 3, x[n] = (\frac{-1}{2})^n u[n]$

$$\begin{aligned}
n \geq 0 & \quad \text{natural: characteristic equation} \\
r - \frac{1}{2} &= 0 \\
y^{(n)}[n] &= c\left(\frac{1}{2}\right)^n \\
& \quad \text{particular} \\
y^{(p)}[n] &= k\left(-\frac{1}{2}\right)^n u[n] \\
k\left(-\frac{1}{2}\right)^n - \frac{1}{2}k\left(-\frac{1}{2}\right)^{n-1} &= 2\left(-\frac{1}{2}\right)^n \\
k &= 1 \\
y^{(p)}[n] &= \left(-\frac{1}{2}\right)^n u[n] \\
& \quad \text{Translate initial conditions} \\
y[n] &= \frac{1}{2}y[n-1] + 2x[n] \\
y[0] &= \frac{1}{2}3 + 2 = \frac{7}{2} \\
y[n] &= \left(-\frac{1}{2}\right)^n u[n] + c\left(\frac{1}{2}\right)^n u[n] \\
\frac{7}{2} &= 1 + c \\
c &= \frac{5}{2} \\
y[n] &= \left(-\frac{1}{2}\right)^n u[n] + \frac{5}{2}\left(\frac{1}{2}\right)^n u[n]
\end{aligned}$$

3.56. Determine the appropriate Fourier representation for the following time-domain signals, using the defining equations.

(a) $x(t) = e^{-t}\cos(2\pi t)u(t)$

Continuous and Nonperiodic, use FT.

$$\begin{aligned} X(j\omega) &= \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt \\ &= \frac{1}{2}\int_{0}^{\infty} e^{-t}(e^{j2\pi t} + e^{-j2\pi t})e^{-j\omega t}\,dt \\ &= \frac{1}{2}\int_{0}^{\infty} e^{-t(1-j2\pi+j\omega)}\,dt + \frac{1}{2}\int_{0}^{\infty} e^{-t(1+j2\pi+j\omega)}\,dt \\ &= \frac{1}{2}\left[\frac{1}{1-j(2\pi-\omega)} + \frac{1}{1+j(2\pi+\omega)}\right] \end{aligned}$$

(b) $x[n] = \begin{cases} \cos(\frac{\pi}{10}n) + j\sin(\frac{\pi}{10}n), & |n| < 10 \\ 0, & \text{otherwise} \end{cases}$

Discrete and Nonperiodic, use DTFT.

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega n}$$

and $x[n] = e^{\frac{\pi}{10}n}, |n| < 10$

=> $X(e^{j\Omega}) = \sum_{n=-9}^{9} e^{j(\frac{\pi}{10}-\Omega)n}$

$$= e^{-j9(\frac{\pi}{10}-\Omega)}\frac{1-e^{j19(\frac{\pi}{10}-\Omega)}}{1-e^{j(\frac{\pi}{10}-\Omega)}}$$

(c) $x[n]$ as depicted in Figure P3.56 (a)

Discrete and Periodic, use DTFS.

We know the fundamental period $N = 7$ and the fundamental frequency $\Omega_0 = 2\pi / N$.

So $X[k] = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk\Omega_0 n}$, where $x[0]=1$, $x[3]=1$, $x[4]= -1$, and $x[1]=x[2]=x[4]=x[6]=0$

$$= \frac{1}{7}(1 + e^{-j\frac{6\pi}{7}k} - e^{-j\frac{8\pi}{7}k})$$

(e) $x(t) = |\sin(2\pi t)|$

Continuous and Periodic, use FS.

$T = \frac{1}{2},\ \omega_o = 4\pi$

$$\begin{aligned} X[k] &= 2\int_{0}^{0.5} \frac{e^{j2\pi t} - e^{-j2\pi t}}{2j} e^{-j4\pi kt}\,dt \\ &= -j\int_{0}^{0.5} e^{j2\pi(1-2k)t}\,dt - j\int_{0}^{0.5} e^{-j2\pi(1+2k)t}\,dt \\ &= \frac{1-(-1)^{1-2k}}{2\pi(1-2k)} + \frac{1-(-1)^{(1+2k)}}{2\pi(1+2k)} \end{aligned}$$

```
% ECE351. Solution to Matlab Experiment 2.86

n = 0:3;
% Generate h1[n]
h1 = 0.25*ones(size(n));
% Generate h2[n]
h2 = 0.25*(-1).^n;
% Generate step function as input
u = ones(1, 20);

% Step response for h1
y1 = conv(u, h1);
% Step response for h2
y2 = conv(u, h2);

figure;
stem(0:19, y1(1:20));
figure;
stem(0:19, y2(1:20));
```

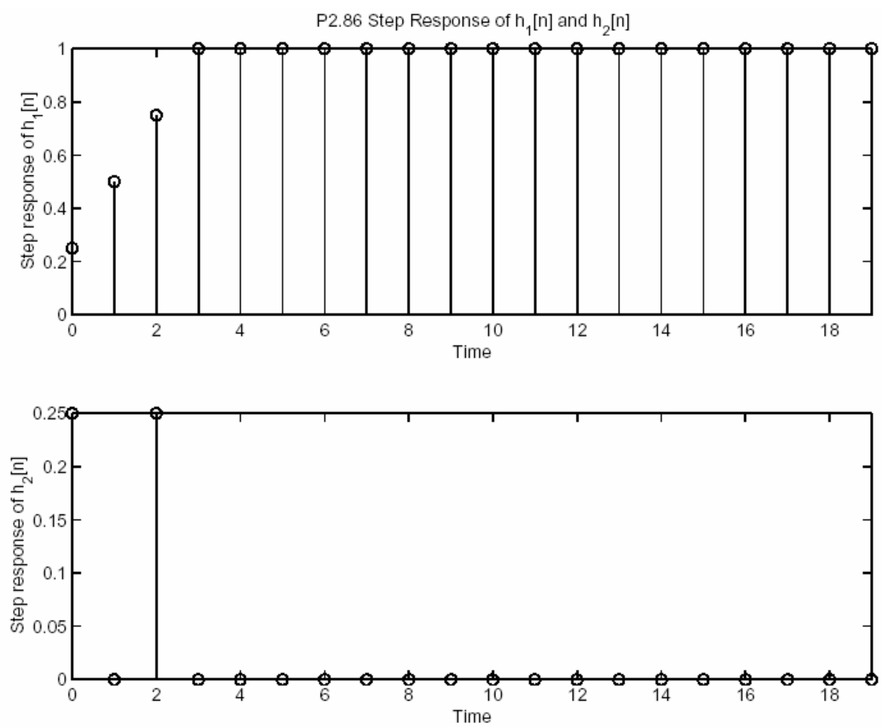

Figure P2.86. Step Response of the two systems

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% ECE351. Solution to Matlab Experiment Part (3)

n = -2:4;
N = 5;
% Generate x[n]
x = zeros(size(n));
x(n==-1) = 0.5;
x(n==0) = 1;
x(n==1) = -0.5;
x(n==4) = 0.5;

% Generate DTFS coefficients
for k = -5:5
   X1(k+6) = sum( x(n>=-2 & n<=2) .* exp(-j*k*(-2:2)*2*pi/N) ) / N;
   X2(k+6) = sum( x(n>=0 & n<=4).*exp(-j*k*(0:4)*2*pi/N) ) / N;
end

X1
X2
```

Output:

```
>> X1 =
  Columns 1 through 4

   0.2000 + 0.0000i   0.2000 + 0.1902i   0.2000 + 0.1176i   0.2000 - 0.1176i

  Columns 5 through 8

   0.2000 - 0.1902i    0.2000             0.2000 + 0.1902i    0.2000 + 0.1176i

  Columns 9 through 11

   0.2000 - 0.1176i   0.2000 - 0.1902i   0.2000 - 0.0000i

>> X2 =
  Columns 1 through 4

   0.2000 - 0.0000i   0.2000 + 0.1902i   0.2000 + 0.1176i   0.2000 - 0.1176i

  Columns 5 through 8

   0.2000 - 0.1902i   0.2000             0.2000 + 0.1902i   0.2000 + 0.1176i

  Columns 9 through 11

   0.2000 - 0.1176i   0.2000 - 0.1902i   0.2000 + 0.0000i
```